Midterm Exam (March 24, 2021): Solve exercise 1 and either exercise 2 or 3.

Exercise 1. Consider an island economy in which there is a single perishable good, consumption. The island is inhabited by two individuals who interact over two dates, today and tomorrow. Mr. 1 is endowed with 10 units of consumption today, and has an insurance policy with a company located in a foreign country compensating him with 20 units of consumption in the event the island suffers a hurricane tomorrow. Ms. 2 is endowed with 10 units of consumption today, and will harvest 20 units of consumption tomorrow unless a hurricane destroys her crops. The preferences for consumption today (x), tomorrow if there is no hurricane (y), and tomorrow if there is a hurricane (z) of both Mr. 1 and Ms. 2 are represented by a utility function of the form $u_i(x,y,z) = x(\alpha_i y + z)$, where $\alpha_1 = 2$ and $\alpha_2 = 1/2$.

(a) (40 points) Assume that the consumers engage in trading contingent contracts. Calculate the (Arrow-Debreu) competitive equilibrium prices and allocation. (Hint. Normalize the price of today consumption to one, i.e., $p_x = 1$. Since $MRS^i_{y,z}(x,y,z) = (\partial u_i/\partial y)\,/\,(\partial u_i/\partial z) = \alpha_i$ (constant), if $1/2 < p_y/p_z < 2$, then consumers' demands satisfy $z_1 = 0$ and $y_2 = 0$. You may guess that CE prices satisfy these inequalities.)

(b) (20 points) Assume instead that the consumers engage in borrowing/lending (denote the market the interest rate by r), insuring one another against the eventuality that a hurricane occurs (denote by q the premium paid today to insure a unit of consumption in that state), and trading in spot markets today and tomorrow (normalize the spot prices to one, i.e., $\hat{p}_x = \hat{p}_y = \hat{p}_z = 1$). Calculate the (Radner) competitive equilibrium interest rate and insurance premium (r^*, q^*). (Hint. There is no need to repeat calculations. Just write the new budget constrains, reducing them to a single equation involving x, y and z. Then you will see the relation between the Arrow-Debreu CE prices, (p_y^*, p_z^*), and the Radner CE interest rate and insurance premium, (r^*, q^*).)

Solution. (a) Since u_i is increasing in y and z, the CE prices satisfy $p_y > 0$ and $p_z > 0$. Also, when $p_y/p_z < 2$ consumers' demand zero units of good, while the supply is positive. Likewise, when $1/2 < p_y/p_z$ consumers' demand zero units of good z, while the supply is positive. Also, it is easy to check that there is no CE in which $1/2 = p_y/p_z$ or $p_y/p_z = 2$. (You may want to prove this.) Hence, in a CE $1/2 < p_y/p_z < 2$, and therefore $z_1 = 0$, $y_2 = 0$.

Therefore (y_1, z_1) satisfies

$$\begin{aligned} \frac{y_1}{x_1} &= \frac{1}{p_y} \\ x_1 + p_y y_1 &= 10 + 20p_z \end{aligned}$$

Solving the system we get

$$x_1(p_y, p_z) = 5 + 10p_z,\ \ y_1(p_y, p_z) = \frac{5 + 10p_z}{p_y}.$$

Likewise (x_2, z_2) *satisfies*

$$\begin{aligned} \frac{z_2}{x_2} &= \frac{1}{p_z} \\ x_2 + p_z z_2 &= 10 + 20p_y. \end{aligned}$$

Solving the system we get

$$x_2(p_y, p_z) = 5 + 10p_y,\ z_2(p_y, p_z) = \frac{5 + 10p_y}{p_z}.$$

Thus, the market clearing conditions are

$$\frac{5 + 10p_z}{p_y} = 20,\ \frac{5 + 10p_y}{p_z} = 20.$$

The solution is $p_y^* = p_z^* = 1/2$. *(Note that* $p_y^*/p_z^* = 1$.*) The CE allocation is* $[(x_1^*, y_1^*, z_1^*), (x_2^*, y_2^*, z_2^*)] = [(10, 20, 0), (10, 0, 20)]$.

(b) In this setting a consumer's budget constraints are

$$\begin{aligned} x &= 10 + b - qv \\ y &= \bar{y} - (1 + r)b \\ z &= \bar{z} - (1 + r)b + v, \end{aligned}$$

where $(\bar{y}, \bar{z})$ *are the consumer's endowments,* b *is the amount s/he borrows and* v *is the number of units of insurance she subscribes. Using the last two equations to solve for* b *and* v*, and substituting in the first equation we may write the budget constraints as the single equation*

$$x + \left(\frac{1}{1 + r} - q\right) y + qz = 10 + \left(\frac{1}{1 + r} - q\right) \bar{y} + q\bar{z}.$$

Since the Radner equilibrium of this economy coincides with the Arrow-Debreu equilibrium, the effective CE prices of consumption in every date and state must be the same, i.e.,

$$\frac{1}{1 + r^*} - q^* = \frac{1}{2},\ q^* = \frac{1}{2}.$$

Thus, $r^* = 0$, $q^* = 1/2$, *and of course the CE allocation is the same as in part (a).*

Exercise 2. A competitive insurance market serves two types of drivers, *inattentive* (h) and *alert* (l), who are present in the population in proportions $\lambda \in (0,1)$ and $1-\lambda$. An inattentive driver has an accident with probability $p_h = 1/2$, while this probability is only $p_l = 1/4$ for alert drivers. All drivers have the same initial wealth, $W = 100$ euros, and their preferences are represented by the Bernuilli utility function $u(x) = \sqrt{x}$. An accident generates a loss of $L = 64$ euros. A driver's type is private information (i.e., not observable).

(a) (10 points) Calculate the full insurance fair policy assuming that all drivers subscribe it. Identify a profitable policy that will attract only alert drivers away from this policy assuming $\lambda = 3/16$.

(b) (20 points) Calculate the separating policy menu, and identify the values of λ for which it is a competitive equilibrium.

(c) (10 points) If there was a majority vote on whether to make mandatory subscribing the pooling policy identified in part (a), for which values of λ will it be approved?

(a) The premium of the fair pooling policy is

$$\bar{I}(\lambda) = \lambda\,(p_h L) + (1-\lambda)\,(p_l L) = 16(1+\lambda).$$

Hence a driver's expected utility is

$$u(100 - \bar{I}(\lambda)) = \sqrt{100 - 16(1+\lambda)}.$$

For $\lambda = 3/13$ the premium of the pooling policy is $\bar{I} = 19$, and the drivers utility is $\bar{u} = 9$. A policy (I, D) satisfying,

$$\begin{aligned} I &> p_l(L-D) \\ (1-p_l)\sqrt{W-I} + p_l\sqrt{W-I-D} &> 9 > (1-p_h)\sqrt{W-I} + p_h\sqrt{W-I-D} \end{aligned}$$

would destabilize the pooling policy. Let us set a sufficiently large deductible, e.g., $D = 12$, and charge a premium above $p_l(L-D) = 13$, e.g., $I = 14$. Then

$$(1-p_l)\sqrt{W-I} + p_l\sqrt{W-I-D} = \frac{3}{4}\sqrt{100-14} + \frac{1}{4}\sqrt{100-14-12} \simeq 9.10,$$

and

$$(1-p_h)\sqrt{W-I} + p_h\sqrt{W-I-D} = \frac{1}{2}\sqrt{100-14} + \frac{1}{2}\sqrt{100-14-12} \simeq 8.93.$$

Hence this policy only attracts low risk drivers, and hence is profitable.

(b) The separating policies are $(I_h, D_h) = (p_h L, 0)$, and (I_l, D_l) satisfying

$$I_l = p_l\,(L - D_l)\,, \text{ and } \frac{1}{2}\sqrt{W - D_l - I_l} + \frac{1}{2}\sqrt{W - I_l} = \sqrt{W - p_h L}.$$

Writing $D_l = x$, and substituting the values in these expressions we may write the second of these equations as

$$\frac{\sqrt{100 - x - \frac{64-x}{4}}}{2} + \frac{\sqrt{100 - \frac{64-x}{4}}}{2} = \sqrt{100-32}.$$

Solving this equation we get

$$\begin{aligned}\tilde{D}_L &= 8\sqrt{17}\left(\sqrt{17}+\sqrt{33}\right)-272 \simeq 53.484,\\ \tilde{I}_L &= \left(64-\left(8\sqrt{17}\left(\sqrt{17}+\sqrt{33}\right)-272\right)\right)/4 \simeq 2.629.\end{aligned}$$

With this policy the expected utility of a low risk driver is

$$\frac{\sqrt{100-53.484-\frac{64-53.484}{4}}}{4}+\frac{3\sqrt{100-\frac{64-53.484}{4}}}{4}=9.057.$$

For this menu to be a CE the policy (I_l, D_l) must be preferred by the low risk drivers to the pooling policy of part (a); that is

$$9.057 \geq \sqrt{100-16(1+\lambda)}.$$

This inequality places a lower bound on $\lambda < \bar{\lambda}$, where

$$\bar{\lambda} = 0.12317.$$

For $\lambda < \bar{\lambda}$ the separating menu is not a CE as it is destabilized by the pooling policy.

(c) Obviously, if $\lambda > 1/2$ the risky drives would be a majority, and they will impose the pooling policy (a), which gives full then insurance for a smaller premium than the separating policy, and hence a greater utility. If $\lambda < \bar{\lambda}$, then both types of drivers prefer the pooling policy (a) to the separating ones, and therefore would unanimously support adopting this regulation. However, for $\lambda \in (\bar{\lambda}, 1/2)$ the prudent drivers form a majority and prefer the separating policy, and therefore the pooling policy (a) would not adopted, and the CE equilibrium separating policies will arise.

Exercise 3. Two fishermen, Art (A) and Bob (B), have free access to a local lake. The total catch of fish (in pounds) obtained by each fisherman depends on how many days a week both fish (z_A, z_B), and is given for $i \in \{A, B\}$ by

$$\frac{8z_i}{3\sqrt{z_A + z_B}}.$$

They have identical preferences for fish and leisure, described by the utility function $u(x, y) = x + y$, where x is the fish consumed, and y is the number of days of leisure during the week. Naturally, each fisherman has 7 days a week for fishing and leisure activities.

(a) (20 points) Calculate how many days will Art and Bob allocate to leisure and fishing. (You may assume that equilibrium is symmetric.) Also, determine whether the allocation is Pareto optimal. (Proof that it is, or identify a Pareto superior allocation.)

(b) (20 points) Identify the socially optimal number of days the two men should fish, and the set of Pareto optimal allocations. Also, calculate (and provide a graph of) the set of possible utilities levels that Art and Bob may enjoy if the reach an optimal individually rational agreement.

Solution. (a) In order to choose z each man solves the problem

$$\max_{z \in [0,7]} \frac{8z}{3\sqrt{z + z_-}} + 7 - z,$$

were z_- is the number of days the other man fishes. The F.O.C. for an interior solution is

$$\frac{24\sqrt{z + z_-} - \dfrac{12z}{\sqrt{z + z_-}}}{9\,(z + z_-)} - 1 = 0.$$

Since the Nash equilibrium is symmetric, i.e., $z = z_-$, denote $w = \sqrt{2z}$. Then the above equation may be written as

$$\frac{24w - \dfrac{6w^2}{w}}{9w^2} = 1,$$

whose solution is

$$w^* = 2.$$

Thus, each men fishes $z^* = 2^2/2 = 2$ days a week for a total catch of fish equal to $8(2)/(3\sqrt{4}) = 8/3$ pounds, and enjoys 5 days of leisure for a total utility equal to $u^* = 23/3 = 7.66$.

This allocation of time is not Pareto optimal: for example, if both men were to reduce their fishing to just one day, then each men total catch of fish would be also $8/(3\sqrt{2})$, and their days of leisure would be 6, for a total utility of $u^* = 8/(3\sqrt{2}) + 6 = 7.89$, which makes both men better off.

(b) Let us identify the total number of fishing days $z \in [0, 14]$ that maximizes social welfare (i.e., the sum of both men utilities). This problem is

$$\max_{z \in [0,14]} \frac{8z}{3\sqrt{z}} + 14 - z.$$

Solving the F.O.C. for an interior solution,

$$\frac{24\sqrt{z} - \dfrac{12z}{\sqrt{z_-}}}{9z} - 1 = 0,$$

we get $z_S = (4/3)^2 \simeq 1.78$ *days, for a total catch of fish of* $32/9 \simeq 3.56$ *pounds of fish. The number of days left for leisure activities of* $14 - (4/3)^2 = 12.22$*. Hence, the set of Pareto optimal allocations is*

$$P = \left\{ [(x_1, y_1), (x_2, y_2)] = [(x, y), \left(32/9 - x, 14 - (4/3)^2 - y\right),\ x \in [0, 32/9],\ y \in [0, 7] \right\}.$$

In an optimal allocation

$$u_A + u_B = (x_1 + y_1) + (x_2 + y_2) = 32/9 + 14 - (4/3)^2 = 14 + (4/3)^2.$$

For an agreement to be individually rational it must give each man the utility he can get if he breaks the agreement and freely decides how many days to fish, which would warrant the utility equal to u^* *calculated in part (a). The graph shows the possible utility levels in an optimal and individually rational agreement.*